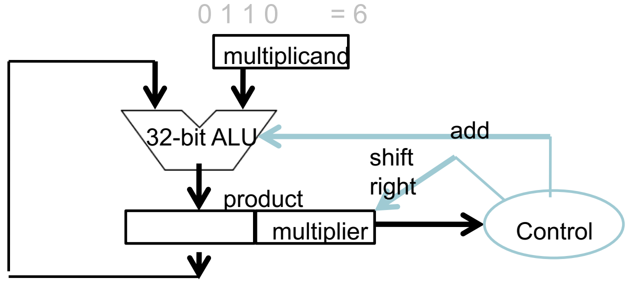
COMPUTER ORGANIZATION AND DESIGN
The Hardware/Software Interface
5th Edition

Chapter 3

Floating Point Arithmetic

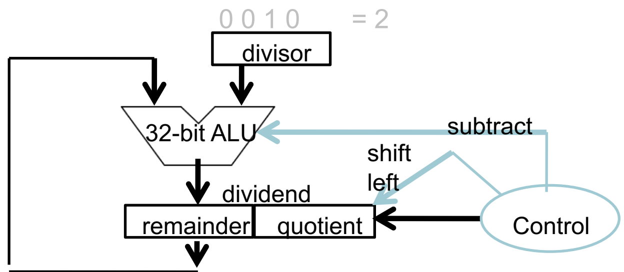
Review - Multiplication

0 1 1 0 = 6




```
     0 0 0 0      0 1 0 1   = 5
add  0 1 1 0      0 1 0 1
     0 0 1 1  →   0 0 1 0
add  0 0 1 1      0 0 1 0
     0 0 0 1  →   1 0 0 1
add  0 1 1 1      1 0 0 1
     0 0 1 1  →   1 1 0 0
add  0 0 1 1      1 1 0 0
     0 0 0 1  →   1 1 1 0   = 30
```

Review - Division



	0 0 0 0		0 1 1 0	= 6
	0 0 0 0	←	1 1 0 0	
sub	1 1 1 0		1 1 0 0	rem neg, so ‘ient bit = 0
	0 0 0 0		1 1 0 0	restore remainder
	0 0 0 1	←	1 0 0 0	
sub	1 1 1 1		1 1 0 0	rem neg, so ‘ient bit = 0
	0 0 0 1		1 0 0 0	restore remainder
	0 0 1 1	←	0 0 0 0	
sub	0 0 0 1		0 0 0 1	rem pos, so ‘ient bit = 1
	0 0 1 0	←	0 0 1 0	
sub	0 0 0 0		0 0 1 1	rem pos, so ‘ient bit = 1

= 3 with 0 remainder

Floating Point

- What can be represented in N bits?
  - Unsigned		0 to 2^N-1
  - 2’s Complement		- 2^{N-1} to 2^{N-1} - 1
- But, what about--
  - Very large numbers?
    - 9,349,398,989,787,762,244,859,087,678
    - 1.23 x 10^{67}
  - Very small numbers?
    - 0.0000000000000000000000045691
    - 2.98 x 10^{-32}
  - Fractional values?		0.35
  - Mixed numbers?		10.28
  - Irrationals?		π

Floating Point

- The essential idea of floating point representation is that a fixed number of bits are used (usually 32 or 64) and that the binary point "floats" to where it is needed. Some of the bits of a floating point representation must be used to say where the binary point lies. The programmer does not need to explicitly keep track of it.
- IEEE (Institute of Electrical and Electronics Engineers) created a standard for floating point. This is the IEEE 754 standard, released in 1985 and updated in 2008. All “main stream” hardware and software follows this standard.

Floating Point

- Floating Point provides representation for non-integral numbers.
- Like scientific notation, we need to “normalize”
  - -2.34×10^{56} ← normalized
  - $+0.002 \times 10^{-4}$ ← not normalized
  - $+987.02 \times 10^{9}$ ← not normalized
- In binary
  - $\pm 1.xxxxxxx_2 \times 2^{yyyy}$
- This is the representation for Types **`float`** and **`double`** in C.

Sign and Magnitude Representation

Sign 1 bit	Exponent 8 bits	Fraction 23 bits
S	E	F

- More exponent bits ➔ wider range of numbers.
- More fraction bits ➔ higher precision.
- For normalized numbers, we are guaranteed that the number is of the form 1.xxxx…. Hence, in IEEE 754 standard, the 1 is implicit.

IEEE Floating-Point Format

	single: 8 bits double: 11 bits	single: 23 bits double: 52 bits
S	Exponent	Fraction

$$x = (-1)^S \times (1 + \text{Fraction}) \times 2^{(\text{Exponent} - \text{Bias})}$$

- S: sign bit (0 ⇒ non-negative, 1 ⇒ negative).
- Normalize significand: 1.0 ≤ |significand| < 2.0
  - Always has a leading pre-binary point 1 bit, so no need to represent it explicitly. This bit is referred to as the “hidden” bit.
  - Significand is the Fraction with the “1.” restored.
- Exponent: excess representation: actual exponent + Bias
  - Ensures exponent is unsigned.
  - Single: Bias = 127; Double: Bias = 1023

Single-Precision Range

- Exponents 00000000 and 11111111 are reserved for exceptions.
- Smallest value
  - Exponent: 00000001
    $\Rightarrow$ actual exponent = 1 – 127 = –126
  - Fraction: 000…00 $\Rightarrow$ significand = 1.0
  - $\pm 1.0 \times 2^{-126} \approx \pm 1.2 \times 10^{-38}$
- Largest value
  - exponent: 11111110
    $\Rightarrow$ actual exponent = 254 – 127 = +127
  - Fraction: 111…11 $\Rightarrow$ significand $\approx$ 2.0
  - $\pm 2.0 \times 2^{+127} \approx \pm 3.4 \times 10^{+38}$

Double-Precision Range

- Exponents 0000…00 and 1111…11 reserved.
- Smallest value
  - Exponent: 00000000001
    $\Rightarrow$ actual exponent = 1 – 1023 = –1022
  - Fraction: 000…00 $\Rightarrow$ significand = 1.0
  - $\pm 1.0 \times 2^{-1022} \approx \pm 2.2 \times 10^{-308}$
- Largest value
  - Exponent: 11111111110
    $\Rightarrow$ actual exponent = 2046 – 1023 = +1023
  - Fraction: 111…11 $\Rightarrow$ significand $\approx$ 2.0
  - $\pm 2.0 \times 2^{+1023} \approx \pm 1.8 \times 10^{+308}$

Floating-Point Example

- What number is represented by the single-precision float 11000000101000…00?
  - S = 1
  - Exponent = $10000001_2 = 129$
  - Fraction = $01000\ldots00_2$
- $x = (-1)^1 \times (1 + 01_2) \times 2^{(129 - 127)}$

  $= (-1) \times 1.25 \times 2^2$

  $= -5.0$

Floating-Point Example

- What is the FP bit representation for -0.75_{10}?
- Represent –0.75
  - $-0.75 = (-1)^1 \times 1.1_2 \times 2^{-1}$
  - S = 1
  - Fraction = $1000\ldots00_2$
  - Exponent = –1 + Bias
    - Single: $-1 + 127 = 126 = 01111110_2$
    - Double: $-1 + 1023 = 1022 = 01111111110_2$
- Single: 1011111101000…00
- Double: 1011111111101000…00

Code Example – Degree Conversion

- MIPS has a second set of 32 32-bit registers reserved for floating point operations.

```
float  f2c (float fahr)
{
    return ((5.0/9.0) * (fahr – 32.0));
}
```

```
(argument fahr is stored in $f12)
 lwc1   $f16, const5($gp)
 lwc1   $f18, const9($gp)
 div.s   $f16, $f16, $f18
 lwc1   $f18, const32($gp)
 sub.s  $f18, $f12, $f18
 mul.s  $f0, $f16, $f18
 jr       $ra
```

Floating-Point Addition

- Addition (and subtraction)

$$(\pm F1 \times 2^{E1}) + (\pm F2 \times 2^{E2}) = \pm F3 \times 2^{E3}$$

  - Step 0: Restore the hidden bit in F1 and in F2.
  - Step 1: Align fractions by right shifting F2 by E1 - E2 positions (assuming E1 $\geq$ E2) keeping track of the lower (or higher) order bits shifted out.
  - Step 2: Add the resulting F2 to F1 to form F3.
  - Step 3: Normalize F3 (so it is in the form 1.XXXXX …)
    - If F1 and F2 have the same sign, shift F3 and increment E3 (check for overflow).
    - If F1 and F2 have different signs, F3 may require *many* left shifts each time decrementing E3 (check for underflow).
  - Step 4: Round F3 and possibly normalize again.
  - Step 5: Rehide the most significant bit of F3 before storing the result.

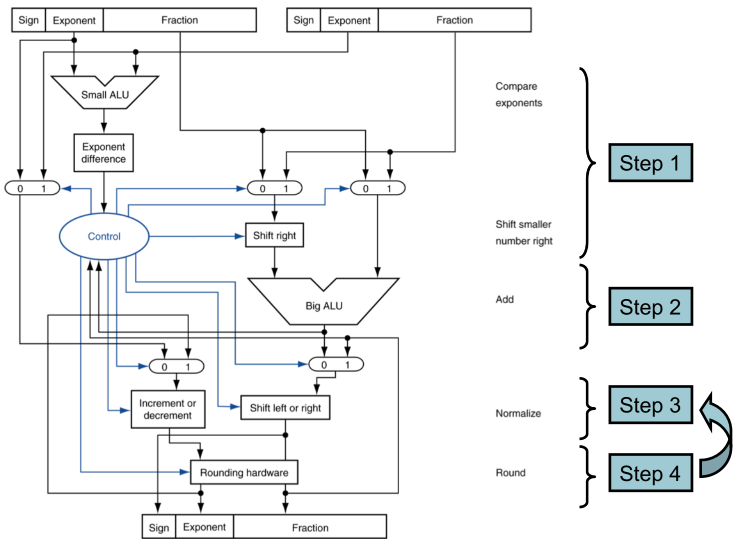
Floating-Point Addition

- Consider a 4-digit decimal example
  - $9.999 \times 10^{1} + 1.610 \times 10^{-1}$
- 1. Align decimal points
  - Shift number with smaller exponent.
  - $9.999 \times 10^{1} + 0.016 \times 10^{1}$
- 2. Add significands
  - $9.999 \times 10^{1} + 0.016 \times 10^{1} = 10.015 \times 10^{1}$
- 3. Normalize result & check for over/underflow
  - 1.0015×10^{2}
- 4. Round and renormalize if necessary
  - 1.002×10^{2}

FP Adder Hardware

- Much more complex than integer adder.
- Doing it in one clock cycle would take too long
  - Much longer than integer operations.
  - Slower clock would penalize all instructions.
- FP adder usually takes several cycles
  - Can be pipelined.

FP Adder Hardware



Floating-Point Multiplication

- Consider a 4-digit decimal example
  - $1.110 \times 10^{10} \times 9.200 \times 10^{-5}$
- 1. Add exponents
  - For biased exponents, subtract bias from sum.
  - New exponent = 10 + –5 = 5
- 2. Multiply significands
  - $1.110 \times 9.200 = 10.212 \Rightarrow 10.212 \times 10^{5}$
- 3. Normalize result & check for over/underflow
  - 1.0212×10^{6}
- 4. Round and renormalize if necessary
  - 1.021×10^{6}
- 5. Determine sign of result from signs of operands
  - $+1.021 \times 10^{6}$

Denormal Numbers

- Exponent = 000...0 $\Rightarrow$ hidden bit is 0

$$x = (-1)^S \times (0 + \text{Fraction}) \times 2^{-\text{Bias}}$$

- Denormal numbers are smaller than normal numbers.
- Denormal with fraction = 000...0

$$x = (-1)^S \times (0 + 0) \times 2^{-\text{Bias}} = \pm 0.0$$

Two representations of 0.0!

Infinities and NaNs

- Exponent = 111...1, Fraction = 000...0
  - ±Infinity
  - Can be used in subsequent calculations, avoiding need for overflow check.
- Exponent = 111...1, Fraction ≠ 000...0
  - Not-a-Number (NaN).
  - Indicates illegal or undefined result
    - e.g., 0.0 / 0.0
  - Can be used in subsequent calculations.

Accurate Arithmetic

- The IEEE 754 Standard specifies additional rounding control
  - Extra bits of precision (guard, round, sticky).
  - Choice of rounding modes.
  - Allows programmer to fine-tune numerical behavior of a computation.
- Not all FP units implement all options
  - Most programming languages and FP libraries just use defaults.
- Trade-off between hardware complexity, performance, and market requirements.
- Rounding (except for truncation) requires the hardware to include extra bits during calculations
  - Guard bit – used to provide one bit when shifting left to normalize a result (e.g., when normalizing after division or subtraction).
  - Round bit – used to improve rounding accuracy.
  - Sticky bit – used to support *Round to nearest even*; it is set to a 1 whenever a 1 bit shifts (right) through it (e.g., when aligning during addition/subtraction).

x86 FP Instructions

Data transfer	Arithmetic	Compare	Transcendental

`FILD mem/ST(i)`
`FISTP mem/ST(i)`
`FLDPI`
`FLD1`
`FLDZ`
	`FIADDP mem/ST(i)`
	`FISUBRP mem/ST(i)`
	`FIMULP mem/ST(i)`
	`FIDIVRP mem/ST(i)`
	`FSQRT`
	`FABS`
	`FRNDINT`
		`FICOMP`
		`FIUCOMP`
		`FSTSW AX/mem`
			`FPATAN`
			`F2XMI`
			`FCOS`
			`FPTAN`
			`FPREM`
			`FPSIN`
			`FYL2X`

- Optional variations
  - I: integer operand.
  - P: pop operand from stack.
  - R: reverse operand order.
  - But not all combinations allowed.

Subword Parallelism

- ALUs are typically designed to perform 64-bit or 128-bit arithmetic.
- Some data types are much smaller, e.g., bytes for pixel RGB values, half-words for audio samples.
- Partitioning the carry-chains within the ALU can convert the 64-bit adder into 4 16-bit adders or 8 8-bit adders.
- A single load can fetch multiple values, and a single add instruction can perform multiple parallel additions, referred to as subword parallelism.

Concluding Remarks

- Bits have no inherent meaning
  - Interpretation depends on the instructions applied.
- Computer representations of numbers
  - Finite range and precision.
  - Need to account for this in programs.
- ISA's support arithmetic
  - Signed and unsigned integers.
  - Floating-point approximation to real numbers.
- Bounded range and precision
  - Operations can overflow and underflow.
- MIPS ISA strongly supports IEEE-754.